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AUTHOR: K. Khandrikh

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ON THE STRUCTURE OF SHOCK WAVES

by

K. Khandrikh

Abstract

The structure of shock waves in a thermally conducting viscous gas is examined. A "law of conservation of irreversible energy flow" within a shock front is derived. Approximate analytic expressions for the integral curve are obtained for the cases where Prandtl's number $P \gg 1$ and $P \ll 1$ and for the velocity profile in a plasma with equal electron and ion temperatures in the approximation $P = 0$.

Introduction

There are numerous methods for determining the structure of a shock wave. However, these numerous results are poorly suited to an investigation of its general properties, since the structure depends on several parameters (for example, the Mach number (M), the ratio of specific heats (γ), Prandtl's number (P)).

Certain general conclusions about the magnitude of the irreversible flows and the profile of a shock wave can be obtained with the help of the "law of conservation of irreversible energy flow," which is presented in the first part of this paper. One can clearly indicate in the "velocity-temperature" plane what contribution anisotropic pressure and heat flow make to the irreversible energy flow.

In the second part of the paper approximate analytic expressions are derived for the integral curve in two cases, $P \gg 1$ and $P \ll 1$. One can ascertain by means of these expressions the accuracy with which the well-known integral curves for $P = \infty$, 0 apply in the case of large and small values of P. They also permit the refinement of the analytic expressions for the integral curve in these cases. Small Prandtl numbers (~ 0.05) occur, for example, in a plasma with equal electron and ion temperatures /2/, large values--(~ 800), for example, in neutral

carbon dioxide gas /3/.

In the last section an approximate analytic expression for the velocity profile is discussed under the assumption of equality of the electron and ion temperatures and with the use of the integral curve for P = 0.

Law of Conservation of Irreversible Energy Flow in a Shock Front

A shock wave, as is well known, is formed as a result of nonlinear effects, which cause an increase in the gradients. The size of these nonlinear effects depends only upon parameters characterizing the equilibrium of the gas and the flow velocity. In stationary shock waves these nonlinear effects, which cause an increase in the gradients, and the irreversible processes, which decrease the gradients, are balanced out.

From the equations of continuity, conservation of momentum and energy for a stationary plane shock wave in an ideal gas (see, for example, /4/), one obtains after simple transformations

$$\gamma u = 1, \quad (1) \qquad -\frac{v_{p_{xx}}}{j^{u}} = \tau - \tau_0, \quad (2) \qquad -\frac{(\gamma-1)q}{j^{u}} = \tau - \tau_\infty, \quad (3)$$

where $$\tau_0 \equiv 1 + (\gamma M^2 u - 1)(1-u), \quad (4)$$

$$\tau_\infty \equiv 1 + (\gamma - 1)\left[\frac{\gamma M^2}{2}(1-u) + 1\right](1-u). \quad j_u \equiv n_a v_u k T_u, \ \tau \equiv \frac{T}{T_a}, \ u \equiv \frac{v}{v_a}, \ \gamma \equiv \frac{n}{n_u}$$

(a denotes a quantity in front of the shock wave), p_{xx} is the anisotropic pressure, q the irreversible heat flux, $M \equiv \left(\frac{mv_u^2}{\gamma k T_u}\right)^{1/2}$, the Mach number, $\gamma \equiv \frac{c_p}{c_v}$, the ratio of specific heats, and the remaining symbols are the usual ones.

If (2) is subtracted from (3), then we get

$$v_{pxx} - (\gamma - 1)q = j_u(\tau_0 - \tau_\infty). \quad (5)$$

With the help of the expression (4) this equation can be rewritten in the form

$$v_{pxx} - (\sigma - 1)q = j_u \frac{2\gamma}{\gamma - 1} M^2 \left(1 - \frac{1}{M^2}\right)^2 (1-z)z, \quad (6)$$

where

$$z \equiv \frac{1-u}{1-u_\beta}, \quad 0 \leqslant z \leqslant 1$$

and $$u_\beta \equiv \frac{v_\beta}{v_u} = \frac{1}{\gamma+1}\left(\gamma - 1 - \frac{2}{M^2}\right).$$

(β denotes a quantity behind the front).

Equation (5) or (6) represents the desired "law of conservation for irreversible processes." The moduli of separate irreversible energy flows contribute to the expression $v_{p_{xx}} - (\gamma - 1)q$, which is responsible for the weakening of the gradients, since it is always the case that $p_{xx} \geq 0$ and $q \leq 0$. On the right-hand side of (5) or (6) there appears the flux caused by the nonlinear effects. It is easy to verify that in the linear approximation in u this flux is equal to zero.

In the stationary case the moduli of both energy flows are equal for any value of the velocity u. From (2), (3), and (5) a simple clear representation in the τ-u plane (Fig. 1) of the irreversible energy flows and the sums of their moduli follows. According to Fig. 1, the integral

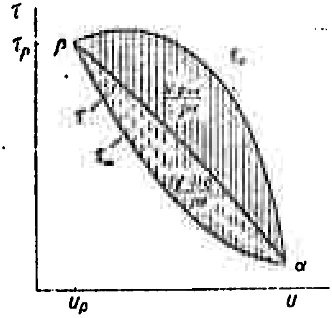

Figure 1.

curve $\tau = \tau(u)$ has no influence on the "sum" of the irreversible energy flows. It only determines what contribution the anisotropic pressure p_{xx} and the heat flux q make.

In order to obtain the velocity and temperature profiles, one must still know the explicit dependence of the transport fluxes on the velocity, the temperature, and their gradients. This dependence has the simplest form in the Navier-Stokes approximation

$$p_{xx} = -\frac{4}{3}\mu(T)\frac{dv}{dx}, \quad q = -\varkappa(T)\frac{dT}{dx}. \tag{7}$$

If the law of conservation of irreversible energy flows (6) is applied, then in particular the following well-known results are easily obtained

with the aid of the Navier-Stokes approximation. In the event of an increase in the transport coefficients μ and $\varkappa$, the gradients $\frac{du}{dx}$ and $\frac{d\tau}{dx}$ must decrease, and consequently the width of the front is increased. Since the conservation law (6) is valid at each point of the shock front, this conclusion is true for any segment of the front. Therefore, it follows that an increase of the transport coefficients with temperature causes the gradients to decrease more and more in the direction of the end of the shock front and the velocity and temperature profiles to be extended. In particular, if the viscosity μ and the thermal conductivity $\varkappa$ depend upon the temperature as T^s, then one can obtain the desired profile from the profile corresponding to $s = 0$ by means of the transformation $dx \to dx' = \tau^s\, dx$.

A similar law of conservation of irreversible energy flows can also be formulated for shock waves in which other irreversible processes, such as radiative exchange, diffusion, etc., play a significant role. For example, for a completely ionized plasma consisting of ions and electrons which are moving with the same velocity, we obtain

$$v(p^i_{xx} + p^e_{xx}) - (\gamma - 1)(q^i + q^e) = j^a \frac{2\gamma}{\gamma - 1} M^2 \left(1 - \frac{1}{M^2}\right)^2 (1 - z)\, z,$$

where p^i_{xx}, q^i, p^e_{xx}, q^e are the corresponding pressure and thermal flux of the ions and the electrons. This equation is also valid in the case where the ion and electron temperatures differ.

Dividing (3) by (2) we get the equation for the integral curve $\tau = \tau(u)$

$$\frac{3}{4} \cdot \frac{1}{P} \cdot \frac{1}{M^2} \cdot \frac{d}{du} = \frac{\tau - \tau_\infty}{\tau - \tau_0}\, u \tag{8}$$

where $P = \frac{\mu}{\varkappa} c_p$ is Prandtl's number* and c_p is the specific heat capacity at constant pressure.

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*According to /5/ one may neglect the dependence of P on the temperature for practically all possible laws of interaction between particles if an ideal gas is considered (c_p = constant).

As is well known, there are three cases in which a simple analytic expression for the integral curve $\tau = \tau(u)$ is obtained, namely for

$P = \infty$, 0, and 3/4 (see, for example, /3/).

It is reasonable to suppose that a small change in the Prandtl number will result in only an insignificant change in the integral curve. Near the points u_α, u_β one can easily prove the validity of this supposition with the aid of the linear approximation. On the basis of these considerations one can attempt to find approximate analytic expression for the integral curve, if P is insignificantly different from the three cases mentioned above, using the known solutions as the zeroth approximations. In the paper by Grad /1/ an approximate method was worked out for P near 3/4. In the present section expressions are derived for the integral curve in the case of large and small P.

P>>1. If we write (8) in the form

$$\tau - \tau_\infty = \frac{3}{4}\frac{1}{P}\frac{1}{M^2}\frac{\tau - \tau_0}{u}\frac{d\tau}{\partial u}, \tag{9}$$

then it is obvious that for sufficiently large Prandtl numbers P one can neglect the right-hand side and thus obtain the zeroth approximation to the integral curve

$$\tau^{(0)} = \tau_\infty. \tag{10}$$

Inserting the zeroth approximation (10) into the right-hand side of (9), we get the first approximation

$$\tau^{(1)} - \tau_\infty = \frac{3}{4}\frac{1}{P}\frac{1}{M^2}\frac{\tau^{(0)} - \tau_0}{u}\frac{d\tau^{(0)}}{du}, \tag{11}$$

and analogously

$$\tau^{(n)} - \tau_\infty = \frac{3}{4}\frac{1}{P}\frac{1}{M^2}\frac{\tau^{(n-1)} - \tau_0}{u}\frac{d\tau^{(n-1)}}{du}. \tag{12}$$

It is necessary for the convergence of the method that all the $\frac{d\tau^{(n)}}{\partial u}$ for $\underline{u}$ in the segment $[u_\alpha, u_\beta]$ remain finite. With the help of equations (4) and (10)-(12), this condition is fulfilled for any values of $\underline{u}$, M, and γ. In the first approximation to the integral curve one obtains in an obvious fashion, for example, the following expression:

$$\tau^{(1)} - \tau_\infty = \frac{3}{4}\frac{1}{P}\frac{1}{M^2}(\gamma - 1)\frac{1-u}{u}[1 + \gamma M^2(1-u)] \times$$
$$\times \left[\gamma M^2 - \frac{\gamma+1}{2}\gamma M^2(1-u) - \gamma\right].$$

<u>P<<1.</u> In this case we write equation (8) in the form

$$\tau - \tau_0 = \frac{4}{3}PM^2u(\tau - \tau_\infty)\frac{1}{\frac{d\tau}{du}}$$

and analogously obtain

$$\tau^{(0)} = \tau_0,$$
$$\tau^{(1)} - \tau_0 = \frac{4}{3} PM^2 u \left(\tau^{(0)} - \tau_\infty\right) \frac{1}{\frac{d\tau^{(0)}}{du}},$$
$$\tau^{(n)} - \tau_0 = \frac{4}{3} PM^2 u \left(\tau^{(n-1)} - \tau_\infty\right) \frac{1}{\frac{d\tau^{(n+1)}}{du}}.$$

In order of this method to converge it is necessary that all the $\frac{d\tau^{(n)}}{du}$ differ form zero. This condition is not always satisfied. It is well known (see, for example, /4/) that under the condition

$$\frac{1}{\gamma}\,\frac{3\gamma-1}{3-\gamma} < M^2 \tag{13}$$

on the interval $[u'_\beta, u_\beta]$ where

$$u_\beta = \frac{1}{\gamma+1}\left(\gamma - 1 - \frac{2}{M^2}\right)$$
$$u'_\beta = \frac{1}{\gamma+1}\left(2 - \frac{\gamma-1}{\gamma M^2}\right)$$

there exists for P = 0 the so-called isothermal jump and $[u_\alpha, u_\beta]$. Therefore we apply the method discussed to the entire range of velocities only under the condition

$$M^2 < \frac{1}{\gamma}\,\frac{3\gamma-1}{3-\gamma} \tag{14}$$

However one can assume that in the opposite case (under condition (13)) the approximations are valid on the half-open segment $[u_\alpha, u_\beta]$. In the first approximation to the integral curve we get

$$\tau^{(1)} - \tau_0 = -\frac{4}{3} PM^2 \frac{1 - \frac{\gamma+1}{2}(1-u) - \frac{1}{M^2}}{2u - 1 - \frac{1}{\gamma M^2}} (1-u)\,u.$$

The detailed analysis of the convergence of the methods discussed is fraught with well-known difficulties. The region and rapidity of convergence depends upon all the parameters u, M, and γ.

In Fig. 2 the first few approximations for P = 10; 1/20; $\gamma = 5/3$; M = 6/5 are shown as an example.

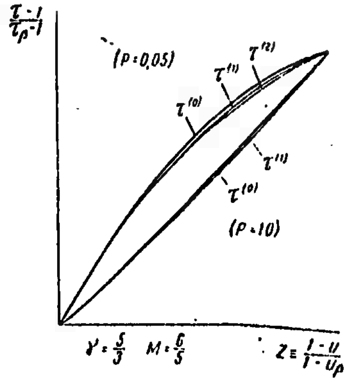

Figure 2.

Velocity Profile in a Plasma

In this section we try to obtain an approximate analytic expression for the velocity profile in a plasma under certain simplifying assumptions. According to /6/ the electron and ion temperatures in a shock wave do not differ very much from one another for $M \leq 2$.* Therefore, in what follows we assume that the electron and ion temperauures are equal. If the relative error of the integral curve is determined with the help of the expression

$$F = \frac{|\tau^{(1)} - \tau^{(0)}|}{\tau_\beta - 1},$$

then in the case of a plasma with equal electron and ion temperatures ($P \approx 0.05$, /2/) we obtain for example, $F_{max} \approx 3\%$ (see Fig. 2) when $M = 6/5$ and $F_{max} \approx 6\%$ when $M^2 \gg 1$. This means that $\tau = \tau^{(c)} \cong \tau_c$ represents a sufficiently good approximation to the integral curve for a plasma.

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*In this case one can also neglect the influence of a balancing of the electron and ion temperatures on the structure of the shock wave.

Substituting the Navier-Stokes approximation (7) into (2) for q and $\tau = \tau_c$(3) for the integral curve and noting that $\frac{d\tau}{dx} = \frac{d\tau}{du}\frac{du}{dx}$ and $\varkappa \sim \mu \sim T^{\frac{5}{2}}$, we obtain

$$\xi - \xi_0 = \frac{15}{4}\frac{1}{M}\int_{u_0}^{u} dt \frac{\left[\frac{5}{3}M^2(1-2t)+1\right]\left[\frac{5}{3}M^2(t-t^2)+t\right]^{\frac{5}{2}}}{\frac{5}{3}M^2(5t-4t^2-1)+5(t-1)}, \quad (15)$$

where

$$\xi \equiv \frac{x}{l_a}, \quad l_a = \frac{1}{P} \frac{\frac{4}{3}\mu(T_a)}{n_a\sqrt{mkT_a}}$$

(Pl_α is the mean free path for momentum transport.) It is found that one can calculate the integral in (15) exactly. The simple but tedious calculations result in an expression which we present in an already simplified form, which represents a good approximation for $M \lesssim 2$,

$$\frac{4}{15} M(\xi - \xi_0) = A \ln\left(\frac{as-1}{as+1}\right) - B \ln\left(\frac{s-\sqrt{u_\beta}}{s+\sqrt{u_\beta}}\right) - $$

$$-2\left(1+\frac{1}{a^2}\right)^5 a^5 \left\{\frac{\frac{4}{5}+\frac{5}{4}s^2+\frac{5}{2}s^4+\frac{9}{8}s^6}{(s^2+1)^4} s - \frac{11}{4} \operatorname{arc\,ctg} s\right\} + c(u_0), \tag{16}$$

where

$$A \equiv \frac{a^2-1}{3a^2-5}, \quad B \equiv \frac{1}{2^{11}} \frac{3-a^2}{3a^2-5}\left[\left(3-\frac{1}{a^2}\right)\left(1+\frac{5}{a^2}\right)\right]^{5/2} a^5, \tag{17}$$

$$s \equiv \left(\frac{1+\frac{1}{a^2}}{u} - 1\right)^{\frac{1}{2}} \text{ и } a^2 \equiv \frac{5}{3} M^2.$$

The constant of integration $C(u_o)$ is chosen such that $u = u_o$ at $\xi = \xi_o$, where u_o and ξ_o are arbitrary quantities. For weak shock waves ($M \to 1$) the first two terms on the right-hand side of (16) play the main role, since $A \sim B \sim \frac{1}{M_2 - 1}$. In shock waves of arbitrary strength, the first term makes the main contribution when $u \to u_\alpha$ and the second term when $u \to u_\beta$. The third term is important only for the center of the shock wave. Thus it is evident that with an increase in M the profile is extended to the end of the shock wave (u u), since the second and third terms in (16)$\sim M^5$, whereas the first term is independent of M in the first approximation.

We recall once more that the expression (16) gives the correct profile in the entire range $[u_\alpha, u_\beta]$ only under condition (14). In the opposite case (16) is applicable only in the range $[u_\alpha, u'_\beta)$ and the change $u_\beta \to u'_\beta$ occurs "at a single instant."

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Department of
Molecular Physics